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Dynamical Localization II with an Application to the Almost Mathieu Operator

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Several recent works have established dynamical localization for Schrödinger operators, starting from control on the localization length of their eigenfunctions, in terms of their centers of localization. We provide an alternative way to obtain dynamical localization, without resorting to such a strong condition on the exponential decay of the eigenfunctions. Furthermore, we illustrate our purpose with the almost Mathieu operator, $H_{\theta, \lambda, \omega} = -\Delta + \lambda \cos(2\pi(\theta + x\omega))$, $\lambda \geqslant 15$ and ω with good Diophantine properties. More precisely, for almost all θ, for all $q > 0$, and for all functions $\psi \in \ell^2(\mathbb{Z})$ of compact support, we show that

$$\sup_t \langle e^{-itH_{\theta, \lambda, \omega}} \psi, |X|^q e^{-itH_{\theta, \lambda, \omega}} \psi \rangle < C_\psi$$

The proof applies equally well to discrete and continuous random Hamiltonians. In all cases, it uses as input a repulsion principle of singular boxes, supplied in the random case by the multi-scale analysis.

KEY WORDS: Dynamical localization; random Schrödinger operator; almost Mathieu model; multiscale analysis; uniform exponential localization.

1. INTRODUCTION

In this article we carry on with the investigation [7, 12] of ways to prove dynamical localization for Schrödinger operators, using spectral hypotheses. Theorem 1.3 below yields dynamical localization for a large class of Schrödinger operators, under a weak uniform exponential localization criterion, that we will call WULE (see Definition 1.2), and that is weaker than the one used in [7] and [12], called SULE (see condition (1.3)).

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Moreover we will show here in detail, in the case of the almost Mathieu operator, how to derive WULE from a repulsion principle of singular clusters essentially proved in [14]. We point out that in the setting of random Schrödinger operators the repulsion principle we use follows precisely from the multi-scale analysis performed by Von Dreifus and Klein in [8], so that a similar reasoning proves WULE [11] for the large class of random Schrödinger operators for which this multi-scale argument has been developed ([4, 21, 8, 12], and references therein).

Let's consider a self adjoint operator H acting on the discrete Hilbert space $\mathscr{H} = \ell^2(\mathbb{Z}^\nu)$ (the same discussion is valid on $L^2(\mathbb{R}^\nu)$).

Definition 1.1. We say that H is dynamically localized on a set of energies I iff for all $q > 0$, and for any initial state ψ with compact support, there exists a constant C_ψ (depending also on I and q but not on t) so that

$$r^{(q)}_{\psi, I}(t) \equiv \langle P_I(H)\, e^{-iHt}\psi, |X|^q\, P_I(H)\, e^{-iHt}\psi \rangle < C_\psi, \qquad \forall t \in \mathbb{R} \tag{1.1}$$

where we denoted by $P_I(H)$ the spectral projector of H onto the set I, and by X the usual position operator (We'll omit the index I if $\sigma(H) \subset I$).

Although, as a result of the RAGE theorem, the condition (1.1) always implies the absence of continuous spectrum, the converse is not true [7]: one needs to add some extra requirements on the eigenfunctions in order to deduce dynamical localization. A first criterion to try is the exponential decay of the eigenfunctions, also called Anderson localization [19], namely: we'll say that a (discrete) Hamiltonian H is *exponentially localized* on I, if its spectrum is pure point on I, and if there exists a $\gamma > 0$ and for each eigenfunction φ_E, $E \in I$, a constant $C(E)$ and a "center of localization" x_E such that

$$|\varphi_E(x)| \leqslant C(E)\, e^{-\gamma\, |x - x_E|}, \qquad \forall x \in \mathbb{Z}^\nu. \tag{1.2}$$

It is now known that exponential localization alone is not sufficient to entail dynamical localization [7]. This is because of the lack of control on the localization length of the eigenfunctions,

$$L(E) = \frac{1}{\gamma} \ln C(E)$$

Consequently the authors of [7] proposed to strengthen this condition by asking for an explicit control of $L(E)$ in terms of centers of localization x_E, namely:

$$C(E) \leqslant C_\varepsilon \exp(\varepsilon\, |x_E|) \tag{1.3}$$

for all $\varepsilon > 0$. They called this condition SULE (Semi-Uniformly Localized Eigenfunctions). SULE does indeed imply dynamical localization, in the discrete case [7] as well as in the continuous case [12]. Moreover SULE has been proven to hold for a large class of random Schrödinger operators: in [7], as an immediate consequence of the results of Aizenman [1], in [12] using a multi-scale argument; and more recently for a quasi-periodic Schrödinger operator [17]. So much for a short review of our subject.

Theorem 1.3 below (proved in Section 2) supplies an alternative way to obtain dynamical localization, starting from a weaker condition than SULE, called WULE below (Weakly Uniformly Localized Eigenfunctions). WULE has the advantage of not asking for an explicit control of the constant $C(E)$ of (1.2) in terms of x_E, as SULE does. It is defined as follows:

Definition 1.2. We say that a Schrödinger operator H has WULE, iff there exist a complete set orthonormal eigenfunctions, φ_E, of H, and constants $\gamma > 0$, $C(l)$, $l \in \mathbb{Z}^\nu$, independent of E, such that, denoting by B the multiplication operator by $b(x) = (1 + x^2)^{-\delta/2}$, $\delta > \nu/2$, one has

$$|\varphi_E(x)\, \varphi_E(l)| \leqslant C(l)\, \|B\varphi_E\|^2_{\ell^2}\, e^{-\gamma\, |x-l|/2}, \qquad \forall x, \quad l \in \mathbb{Z}^\nu, \quad \forall E \tag{1.4}$$

or equivalently, with $\widetilde{\varphi_E} = \varphi_E / \|B\varphi_E\|_{\ell^2}$,

$$|\,\widetilde{\varphi_E}(x)\ \widetilde{\varphi_E}(l)| \leqslant C(l)\, e^{-\gamma\, |x-l|/2}, \qquad \forall x, \quad l \in \mathbb{Z}^\nu, \quad \forall E. \tag{1.5}$$

We first briefly mention that WULE is trivially stronger than the exponential decay of the eigenfunctions (condition (1.2)). On the other hand, as already claimed, it is weaker than SULE, and we provide a proof of this point in the appendix.

The first theorem relates WULE to the dynamical localization of the operator H:

Theorem 1.3. Let H be a self-adjoint operator on $\ell^2(\mathbb{Z}^\nu)$ and suppose that H has WULE. Then, for all ψ with $\operatorname{supp} \psi \subset [-R, R]^\nu$, $R > 0$, and for some constant $C_R = C(R, q, \gamma, \delta)$, one has

$$\sup_t r_\psi^{(q)}(t) \leqslant C_R\, \|\psi\|^2_{\ell^2}$$

i.e., H is dynamically localized.

Let us mention that a local version of both Definition 1.2 and Theorem 1.3 (i.e., restricting the energies to a compact interval I) is obviously available. Moreover since the almost Mathieu operator is a discrete operator, Definition 1.2 and Theorem 1.3 have been stated in their discrete version. Nevertheless, we will provide a proof of this theorem for continuous Schrödinger operators at the end of Section 2, together with a continuous analog of the condition (1.4).

Using WULE, rather than SULE, in order to obtain dynamical localization has two advantages: first, *because* WULE requires no control in terms of centers of localization, it is less technical to establish than SULE (see remarks in Section 3); next, although WULE is weaker than SULE, it turns out surprisingly that it is easier to prove dynamical localization starting from WULE, rather than from SULE (see proofs in Section 2). But as a limitation of WULE, in comparison to SULE, it should be pointed out that WULE only assures the non-spread of compactly supported initial states, while SULE allows exponentially fast decay of the initial wave-packets [7, 12]: this is the price one has to pay for the simplifications mentioned above.

As already said, WULE holds for all random or quasi-periodic Schrödinger operators for which a multi-scale type argument has been developped in order to establish the exponential localization of the eigenfunctions [11].

Proofs of SULE and WULE can be divided in two parts. The first one, which is common to both, is a kind of repulsion principle of two singular boxes, which is the main and well-kown result of the multi-scale analysis performed by Von Dreifus and Klein in [8] (Theorem 2.2 in [8]) and which has been obtained for the almost Mathieu model by Jitomirskaya [14] (Lemma 3.3 below). The main difference appears in the second part, i.e., in the way one uses the fruits of the first part: instead of introducing the centers of localization (to get SULE), we apply (to get WULE) some ideas of Jona-Lasinio, Martinelli and Scoppola [18]. This is what is done in Section 3 below.

Before turning to the almost Mathieu model and to Theorem 1.5, let us finish with a last remark on WULE. It is important to mention that WULE provides the same semi-stability as SULE, under rank one perturbations. Indeed, it is not hard to check that Theorem 8.1 of [7] is still true if one supposes only WULE instead of SULE. Hence, if WULE holds, the momenta $r_{\delta_0}^{(q)}(t)$ associated to the discrete perturbed Hamiltonian $H+\xi\langle\cdot,\delta_0\rangle\,\delta_0$ can not grow faster in t than logarithmically. This is the content of Theorem 1.4, that derives directly from Theorem 8.1 of [7] and Proposition 2.1 below.

Theorem 1.4. Let H be a self-adjoint operator on $\ell^2(\mathbb{Z}^\nu)$ with pure point spectrum, and $H_\xi = H+\xi\langle\cdot,\delta_0\rangle\,\delta_0$. Suppose that H has WULE. Then one has, for some constant C,

$$\langle e^{-iH_\xi t}\delta_0, |X|^q\, e^{-iH_\xi t}\delta_0\rangle < C(\mathrm{Log}\,|t|)^q.$$

Let us turn to the application we have chosen in order to illustrate how WULE can be obtained from the result of the multi-scale analysis. The almost Mathieu operator is defined on $\ell^2(\mathbb{Z})$ by

$$(H_{\theta,\lambda,\omega}u)(x) = u(x-1) + u(x+1) + \lambda\cos(2\pi(\theta + x\omega))\,u(x), \qquad x \in \mathbb{Z}, \tag{1.6}$$

with $\lambda > 0$, $\omega \in \mathbb{R}\backslash\mathbb{Q}$, and $\theta \in [0, 1[$. We shall need the following definition too. We say that an irrational number ω has Diophantine properties of order $r > 2$, if for some $C > 0$, p_n/q_n, being the n^{th} continuous fraction approximant of ω, one has

$$\left|\omega - \frac{p_n}{q_n}\right| > \frac{C}{q_n^r}. \tag{1.7}$$

Let us recall that the spectrum of these operators has been extensively studied and offers an interesting diversity that depends on the parameter ω (for complements and proofs we refer to [5, 13, 15, 21]). If ω is rational, then the potential is periodic and the spectrum is absolutely continuous. In that case, it is known that $r^{(2)}(t) \sim Ct^2$. When $\lambda > 2$, if ω is irrational but extremely well approximated by rational numbers (Liouville numbers), the spectrum is singular continuous ([2] using a theorem of Gordon, see also [5]). On the dynamical side, one then only knows that, at least for a dense G_δ set of ω, $\limsup_{t\to\infty} r_\psi^{(q)}(t)\, t^{-(2-\varepsilon)} = \infty$, $\forall \varepsilon > 0$ (for a dense set of ψ) [20, 6]. Finally, and this is the case we are interested in, if ω is situated "sufficiently far" from $\mathbb{Q}$ (good Diophantine properties), then, for a.e. θ, the spectrum is pure point, at least if $\lambda \geqslant 15$. This result, together with the exponential localization of the eigenfunctions, has been proved for λ large enough in [23], and in [10, 24] (using an adaptation of the multi-scale analysis originally developed for the Anderson model in [9]). More recently this result has been substantially improved by S. Jitomirskaya in [14] and [15], using the fruitful ideas of Theorem 2.3 of [8].

We will prove in Section 3 the following statement:

Theorem 1.5. Let $\lambda \geqslant 15$ and suppose ω satisfies (1.7) with $r > 2$. There exists a set Θ (known explicitly) with full Lebesgue measure in $[0, 1]$, such that:

(i) H_θ has WULE,

(ii) for all $\theta \in \Theta$, $q > 0$, and for all $\psi \in \ell^2(\mathbb{Z})$ with compact support, there exists a constant $C(\psi, \theta, q)$ so that

$$\sup_t r_\psi^{(q)}(t) < C(\psi, \theta, q).$$

As already mentioned Jitomirskaya and Last [17] proved that SULE holds under the same assumptions, which is a stronger result. Nevertheless, if one is only interested in the dynamical part of the result in the sense of Definition 1.1, then Theorem 1.5 supplies an alternative and simpler proof.

Remark 1. Ref. [14] tells us that the result also holds for smaller values of λ ($\lambda > 5.4$), but only for a restricted interval $I = \sigma_{pp}(H_{\theta, \lambda, \omega}) \cap [-\varepsilon(\lambda), \varepsilon(\lambda)]$, and no longer for the whole spectrum.

Remark 2. One would like to extend this result to smaller values of λ ($\lambda \in]2, 15[$), where pure point spectrum has been proven to be of full measure (with the exponential decay of the corresponding eigenfunctions), but without ruling out the possibility of a singular continuous component of the spectrum (of zero Lebesgue measure) [15]. In fact, to extend this result, one would need a version of the essential Lemma 3.3 for such λ.

2. DYNAMICAL LOCALIZATION

This section deals with the dynamical part of this article, and hence the proof of Theorem 1.3. In [7], it is shown that SULE is closely related to another criterion, written in a more dynamical way, called SUDL (Semi-uniformly Dynamical Localization). The same phenomenon appears with WULE, as can be seen, comparing (1.5) and (2.1), with the following proposition:

Proposition 2.1. Let $H = -\Delta + V$ be a Schrödinger operator defined on $\ell^2(\mathbb{Z}^\nu)$, $\nu \geqslant 1$. Suppose that H has WULE, then there exist $\gamma > 0$, and for all $l \in \mathbb{Z}^\nu$ a constant $C'(l)$ so that

$$|\langle \delta_x, e^{-iHt} \delta_l \rangle| \leqslant C'(l)\, e^{-\gamma\, |x-l|}, \qquad \forall x, \quad l \in \mathbb{Z}^\nu, \quad \forall t \in \mathbb{R}. \tag{2.1}$$

This proposition is the main step in the proof of Theorem 1.3. Like SUDL in [7], the estimate (2.1) appears therefore to be the key point that supplies dynamical localization (as stressed in [3]). Note that the exponential decay of the two-point function $\langle \delta_x, e^{-iHt} \delta_l \rangle$ has already been introduced in the past as a dynamically relevant criterion [1, 3, 16, 17, 19].

It is only for a question of readability that we assumed, in the definition of WULE, the spectrum is pure point. This also makes the proof of Proposition 2.1 shorter. Anyhow Proposition 2.1 remains valid if the Hamiltonian is not known *a priori* to have pure point spectrum. One then has to resort explicitly to an eigenfunctions expansion formula [4, 21, 22], and to work with the uniformly polynomially bounded eigenfunctions $\widetilde{\varphi_E}$. Of course, pure point spectrum is then a consequence of the result.

Proof of Proposition 2.1. Let us first recall that B denotes the multiplication operator by $b(x) = (1 + |x|^2)^{-\delta/2}$, $\delta > \nu/2$. One easily checks that

$$\sum_E \|B\varphi_E\|_{\ell^2}^2 = \sum_{x \in \mathbb{Z}^\nu} b(x)^2 \sum_{E \text{ eigenvalue}} |\varphi_E(x)|^2 = \|b\|_{\ell^2}^2 < \infty \tag{2.2}$$

In fact, this quantity is nothing but the mass $\rho(\mathbb{R})$ of the spectral measure $\rho(\Delta) = \mathrm{tr}(BE(\Delta)\, B)$ that appears in the eigenfunctions expansion formula [22]. Now, one easily obtains:

$$\begin{aligned} |\langle \delta_x, e^{-iHt}\delta_l\rangle| &\leqslant \sum_E |\varphi_E(x)\, \varphi_E(l)| \\ &\leqslant \left(\sum_E \|B\varphi_E\|_{\ell^2}^2\right) C(l)\, e^{-\gamma\,|x-l|/2} \\ &\leqslant \|b\|_{\ell^2}^2\, C(l)\, e^{-\gamma\,|x-l|/2} \end{aligned}$$

and the claimed implication (WULE) $\Rightarrow$ (2.1) is proved. ∎

We now deduce dynamical localization.

Proof of Theorem 1.3.

$$\begin{aligned} \||X|^{q/2}\, e^{-iHt}\delta_l\|_{\ell^2}^2 &\leqslant \sum_{x \in \mathbb{Z}^\nu} |x|^q\, |e^{-iHt}\, \delta_l(x)|^2 \\ &\leqslant C(l, \delta, q) \sum_{x \in \mathbb{Z}^\nu} (|l|^q + |x-l|^q)\, e^{-\gamma\,|x-l|} \\ &\leqslant C(l, q, \gamma, \delta) \end{aligned} \tag{2.3}$$

where we used the following trivial fact: if $p \geqslant 1$, one has, for $a, b > 0$,

$$(a+b)^p \leqslant 2^{p-1}(a^p + b^p) \tag{2.4}$$

Finally, taking $\psi = \sum_{|l| \leqslant R} \psi(l)\, \delta_l$, and using again (2.4):

$$\begin{aligned} (r_\psi^{(q)}(t))^{1/2} &= \||X|^{q/2}\, e^{-iHt}\psi\|_{\ell^2} \\ &\leqslant \sum_{|l| \leqslant R} |\psi(l)|\, \||X|^{q/2}\, e^{-iHt}\delta\|_{\ell^2} \\ &\leqslant C(R, q, \gamma, \delta)\, \|\psi\|_{\ell^2}. \quad ∎ \end{aligned}$$

The Continuous Case. We want here to provide an analog of WULE and of Proposition 2.1 for the continuous case. Consider $H=-\Delta+V$, a Schrödinger operator, with V a potential that belongs to the Kato class K^{ν} [5, 22]; and suppose that the following analog of the estimate (1.4) holds, uniformly in $E\in I$ a compact interval:

$$\|\chi_x\varphi_E\|_{\mathrm{L}^2}\,\|\chi_l\varphi_E\|_{\mathrm{L}^2}\leqslant C(l)\,\|B\varphi_E\|_{\mathrm{L}^2}^2\,e^{-\gamma\,|x-l|},\qquad \forall x\in\mathbb{R}^{\nu},\quad \forall l\in\mathbb{Z}^{\nu}\tag{2.5}$$

where χ_l is the characteristic function of a box $\Lambda(l)$ of size $r>0$ centered at point l. We then claim that H is dynamically localized on I.

Indeed, take $\psi\in\mathrm{L}^2(\mathbb{R}^{\nu})$ with compact support, let's say $[-R,R]^{\nu}$; and denote by $\Gamma_{r,R}$ the lattice $r\mathbb{Z}^{\nu}\cap[-R,R]^{\nu}$; one has:

$$\begin{aligned}
&\||X|^{q/2}\,e^{-iHt}P_I(H)\,\psi\|_{\mathrm{L}^2}^2\\
&\quad\leqslant\sum_{E\in I}|\langle\psi,\varphi_E\rangle|\,\||X|^q\,\varphi_E\|_{\mathrm{L}^2}\,\|\psi\|_{\mathrm{L}^2}\\
&\quad\leqslant\sum_{E\in I}\sum_{l\in\Gamma_{r,R}}\|\chi_l\psi\|_{\mathrm{L}^2}\,\||X|^q\,\varphi_E\|_{\mathrm{L}^2}\,\|\psi\|_{\mathrm{L}^2}\\
&\quad\leqslant\sum_{E\in I}\sum_{l\in\Gamma_{r,R}}\|\chi_l\psi\|_{\mathrm{L}^2}\,\|\psi\|_{\mathrm{L}^2}\left(\int dx\,|x|^{2q}\,\|\chi_l\varphi_E\|_{\mathrm{L}^2}^2\,\|\chi_x\varphi_E\|_{\mathrm{L}^2}^2\right)^{1/2}\\
&\quad\leqslant\left(\sum_{E\in I}\|B\varphi_E\|_{\mathrm{L}^2}^2\right)\|\psi\|_{\mathrm{L}^2}\sum_{l\in\Gamma_{r,R}}\|\chi_l\psi\|_{\mathrm{L}^2}\,C(l)\left(\int|x|^{2q}\,e^{-2\gamma\,|x-l|}\,dx\right)^{1/2}\\
&\quad\leqslant C(R,r,\gamma,q)\,\mathrm{tr}(BP_I(H)\,B)\,\|\psi\|_{\mathrm{L}^2}^2
\end{aligned}$$

which is bounded since, considering $V\in K^{\nu}$, $BP_I(H)\,\mathrm{B}$ is a trace class operator [22]. Note that this operator remains trace class if one adds a magnetic field (see [22] for precise hypotheses), so that this result of dynamical localization, assuming (2.5), applies to the continuous Anderson or Landau operators we were interested in in [12], i.e., with a random potential, i.i.d, and of bounded probability density.

3. PROOF OF THEOREM 1.5

Let us recall that the HamiltoFlian we consider here is defined on $\ell^2(\mathbb{Z})$ and is given by (1.6). We first need some definitions and results. We denote by $H_{[x,y],\theta}$ the restriction of H_θ to the interval $[x,y]$ with zero boundary conditions at the points $x-1$ and $y+1$, and by $G_{[x,y]}(E)$ the Green function $(H_{[x,y],\theta}-E)^{-1}$ defined on $\mathbb{R}\backslash\sigma(H_{[x,y],\theta})$ (note that we'll drop the θ-dependence of the Green function). Following the usual vocabulary coming from the multi-scale analysis, let's define the notions of "regularity" and "singularity" in our case [14]:

Definition 3.1. Lef $\gamma > 0$, $k > 0$, and $E \notin \sigma(H_{[x,y],\theta})$. A point $z \in \mathbb{Z}$ is called (γ, k)-regular at energy E if there exists an interval $[x, y]$, with $|x-y| \leqslant k$, containing z and such that

$$|G_{[x,y]}(E, z, u)| < e^{-\gamma k/2}, \qquad u = x, y.$$

Otherwise z is called (γ, k)-singular at energy E.

We recall the following well known identity. Let $z \in [x, y]$, $E \notin \sigma(H_{[x,y],\theta})$, and $\varphi \in \ell^2(\mathbb{Z})$ such that $H_\theta \varphi = E\varphi$. One then has

$$\varphi(z) = G_{[x,y]}(E, x, z)\, \varphi(x-1) + G_{[x,y]}(E, z, y)\, \varphi(y+1) \tag{3.1}$$

We now state the two lemmas that will entail the result. The first lemma is a standard consequence of identity (3.1), and its proof is given below.

Lemma 3.2. Let $z \in \mathbb{Z}$ and $\gamma > 0$. Suppose that φ_E is a polynomially bounded solution of $H_\theta \varphi = E\varphi$ and that $\varphi_E(z) \neq 0$. Then, there exists a constant $k_1(E, \theta, z) > 0$ such that the point z is (i) (γ, k)-singular for all $k > k_1(E, \theta, z)$ and (ii) $(\gamma, k_1(E, \theta, z))$-regular if $k_1(E, \theta, z) > 1$.

The second Lemma is an analog of Theorem 2.2 in [8], and provides what has been called above a repulsion principle of singular boxes.

Lemma 3.3. Let $l \in \mathbb{Z}$. For a.e. θ, there exist $k_2(\theta, l)$ (independent of E) and $\gamma > 0$ such that, if $|x - l| = k > k_2(\theta, l)$ then for all $E \in I$, the points x and l can't be simultaneously (γ, k)-singular.

Jitomirskaya gave a proof of this Lemma 3.3 in [14], but without taking care of the E-dependence of each step. This E-independence, as also pointed out in [17], is the main ingredient that provides a dynamical result. This lemma is then common to both the proof of SULE [12, 17] and the one of WULE. As already noticed, the difference appears only after this step, in the way one exploits the fruits of this lemma. Because of the extra (compared to WULE) control in terms of centers x_E required by SULE, it is less technical to derive WULE from Lemma 3.3, rather than SULE.

More precisely, once this lemma is obtained, the way the dependence in energy of the different parameters is controlled differs depending on whether one turns toward SULE or WULE. To get WULE, the point where appears the energy parameter no longer lies in the use of centers of localization x_E, as in SULE, but in the trivial Lemma 3.2, a fraternal twin of Lemma 3.5 in [12]: this is why obtaining WULE is simpler than proving SULE.

Concerning the proof of Lemma 3.3, a careful reading of [14], paying attention to the dependence in E of the constants, together with the technical Lemma 4.5 in [17], shows that $k_2(\theta, l)$ is uniform in E, as claimed in Lemma 3.3 above. We stress that although this careful reading is not just a trivial check, it doesn't require any further arguments than the one already developed in [14] by Jitomirskaya. We thus state Lemma 3.3 without proof. Anyhow we mention that one can also find in [17], the E-independent versions of the different lemmas contained in [14] that lead to Lemma 3.3.

Proof of Lemma 3.2. Take φ_E, z and γ as in the lemma. Suppose z is (γ, k)-regular for all $k>0$. Then, with $x=z-[k/2]$ and $y=z+[k/2]$, Relation (3.1) gives us, for all $k>0$,

$$|\varphi_E(z)| \leqslant p_z(k)\, e^{-\gamma k/2},$$

where $p_z(k)$ is a polynomial in k, which is impossible, since $\varphi_E(z) \neq 0$. In addition, if the smallest integer where the above reasoning fails is strictly bigger than 1, one has the second point of the lemma. ∎

Proof of Theorem 1.5. The lines of the proof can be sketched as follows. From Lemma 3.2 and Lemma 3.3 stated above, we derive the first part of the theorem, i. e. WULE. More precisely, we establish the condition (1.5) that deals with the uniformly polynomially bounded eigenfunctions $\widetilde{\varphi_E}$ (rather than the condition (1.4) that concerns the normalized eigenfunctions φ_E). Then Theorem 1.3 gives us the announced result.

The spectrum of H_θ being pure point [14] (for θ, ω and λ as in the theorem), let φ_E be the eigenfunctions of H_θ; and consider, for $\delta > 1/2$, the new eigenfunction $\widetilde{\varphi_E} = \varphi_E / \|B\varphi_E\|_{\ell^2}$. Remark that, since $\|B\,\widetilde{\varphi_E}\|_{\ell^2} = 1$, $\widetilde{\varphi_E}$ trivially satisfies the following essential E-independent bound: for all $x \in \mathbb{Z}^\nu$, and for all eigenvalues E:

$$|\,\widetilde{\varphi_E}(x)| \leqslant (1+|x|)^\delta. \tag{3.2}$$

We stress that the independence in E of the right hand side of (3.2) is crucial in order to get the estimate (1.5) uniformly in E.

The aim is then to prove the estimate (1.5), i.e. that for all $x, l \in \mathbb{Z}$,

$$|\,\widetilde{\varphi_E}(x)\ \widetilde{\varphi_E}(l)| \leqslant C(l)\, e^{-\gamma\,|x-l|/2},$$

for some $\gamma > 0$ independent of $\widetilde{\varphi_E}$. First remark that one can clearly suppose $\widetilde{\varphi_E}(l) \neq 0$. Hence Lemma 3.2 gives us a $k_1(E, \theta, l)$, that we will first suppose bigger than 1: thus the point l is $(\gamma, k_1(E, \theta, l))$-regular, and (γ, k)-singular for all $k > k_1(E, \theta, l)$. Let us define

$$\bar{k}(E, \theta, l) = \max(k_1(E, \theta, l), 1, k_2(\theta, l)).$$

Together with Lemma 3.3, one then obtains that for all $x \in \mathbb{Z}$, one has

$$(|x-l| > \bar{k}(E, \theta, l)) \Rightarrow (x \text{ is } (\gamma, |x-l|)\text{-regular}).$$

Thus, the identity (3.1), combined with Definition 3.1 and the bound (3.2), yields:

$$|\widetilde{\varphi_E}(l)| \leqslant C_1(1 + |l| + k_1(E, \theta, l))^{\delta} e^{\gamma k_1(E, \theta, l)/2} \tag{3.3}$$

and

$$|\widetilde{\varphi_E}(x)| \leqslant C_2(1 + |l| + 2|x-l|)^{\delta} e^{-\gamma |x-l|/2}, \qquad \text{if} \quad |x-l| > \bar{k}(E, \theta, l). \tag{3.4}$$

Suppose the energy E is such that $\bar{k}(E, \theta, l) = k_1(E, \theta, l) > 1$. Notice that this must happen an infinite number of times, since, if not, one would obtain a uniform exponential localization of the eigenfunctions (ULE); and it is known [7, 16] that ULE implies a phase-stability, incompatible with Anderson or even bounded quasi-periodic models (and consequently with the almost Mathieu model).

Using (3.3) in order to control $|\widetilde{\varphi_E}(l)|$ and respectively (3.2) or (3.4) in order to control $|\widetilde{\varphi_E}(x)|$, depending on whether $|x-l| \leqslant k_1(E, \theta, l)$ or $|x-l| > k_1(E, \theta, l)$, one obtains,

$$\begin{aligned} |\widetilde{\varphi_E}(x)\ \widetilde{\varphi_E}(l)| &\leqslant C_3(\delta)(1 + |l| + k_1(E, \theta, l))^{\delta} e^{-\gamma k_1(E, \theta, l)/2} \\ &\times \begin{cases} (1 + |l| + k_1(E, \theta, l))^{\delta} & \text{if} \quad |x-l| \leqslant k_1(E, \theta, l), \\ (1 + |l| + 2|x-l|)^{\delta} e^{-\gamma |x-l|/2}, & \text{if} \quad |x-l| > k_1(E, \theta, l). \end{cases} \end{aligned}$$

At this point one understands the important role of the factor $|\widetilde{\varphi_E}(l)|$, since, thanks to the $(\gamma, k_1(E, \theta, l))$-regularity of the point l, one will be able to control the polynomial growth in $k_1(E, \theta, l)$ in the above estimates.

Pick $0 < \gamma' < \gamma$. It is easy to see that there exists a constant C', just depending on δ, such that

$$(1 + |l| + 2|z|)^{\delta} e^{-\gamma |z|/2} < C'(1 + |l|)^{\delta} e^{-\gamma' |z|/2}, \qquad \forall z \in \mathbb{R}, \quad \forall l \in \mathbb{Z}.$$

Apply this:

$$
|\widetilde{\varphi_E}(x)\ \widetilde{\varphi_E}(l)| \leqslant C_4(\delta)(1+|l|)^{2\delta} \times \begin{cases} e^{-\gamma' k_1(E,\theta,l)/2} & \text{if} \quad |x-l| \leqslant k_1(E,\theta,l) \\ e^{-\gamma' k_1(E,\theta,l)/2} e^{-\gamma' |x-l|/2}, & \text{if} \quad |x-l| > k_1(E,\theta,l). \end{cases}
$$

And, in both cases, one has

$$
|\widetilde{\varphi_E}(x)\ \widetilde{\varphi_E}(l)| \leqslant C_4(\delta)(1+|l|)^{2\delta}\, e^{-\gamma' |x-l|/2} \qquad \forall x, \quad l \in \mathbb{Z}.
$$

The proof is now almost finished. We just have to examine the second case, that is if E is such that $\bar{k}(E,\theta,l) = k_2(l,\theta)$ or is equal to 1. In both cases, the crucial point is that $\bar{k}(E,\theta,l)$ doesn't depend any more on E. Hence, using this time (3.2) in order to bound $|\widetilde{\varphi_E}(l)|$, one obtains that for some constant $C_5(l)$, uniform in energy:

$$
|\widetilde{\varphi_E}(x)\ \widetilde{\varphi_E}(l)| \leqslant C_5(l)\, e^{-\gamma' |x-l|/2} \qquad \forall x, \quad l \in \mathbb{Z},
$$

and that ends the proof of Theorem 1.5. ∎

APPENDIX

For the reader's convenience, we show in this appendix how to derive WULE from SULE, in the discrete case.

Proposition A.1. Let H be a discrete Schrödinger operator such that there exist a complete set orthonormal eigenfunctions, φ_E, centers $|x_E|$, and E-independent constants $\gamma > 0$, $\varepsilon < \gamma/3$ and $C_\varepsilon > 0$, so that

$$
|\varphi_E(x)| < C_\varepsilon e^{\varepsilon |x_E|} e^{-\gamma |x - x_E|}, \qquad \forall x \in \mathbb{Z}^\nu, \quad \forall E.
$$

Then H has WULE.

Proof of Proposition A.1. Using the hypothesis of the proposition one easily checks the following:

$$
|\varphi_E(x)\, \varphi_E(l)| \leqslant C_\varepsilon^2 e^{-\varepsilon |x_E|} e^{2\varepsilon |l|} e^{-(\gamma - 3\varepsilon)|x-l|},
$$

with $x, l \in \mathbb{Z}^\nu$. It is then clear that the proposition will be proved if one shows that for some E-independent constant K, one has

$$
e^{-\varepsilon |x_E|} \leqslant K \, \|B\varphi_E\|_{\ell^2}^2 \tag{A.1}
$$

with B the operator defined previously in Definition 1.2. To obtain inequality (A.1), let us observe that

$$\sum_{|x-x_E|>|x_E|} |\varphi_E(x)|^2 \leqslant C_\varepsilon^2 \sum_{|x-x_E|>|x_E|} e^{-2(\gamma-\varepsilon)\,|x-x_E|} \leqslant C(\varepsilon,\gamma)\, e^{-2(\gamma-\varepsilon)\,|x_E|}$$

It therefore follows, since $(|x-x_E|\leqslant|x_E|)\Rightarrow(|x|\leqslant 2\,|x_E|)$:

$$\|B\varphi_E\|_{\ell^2}^2 \geqslant (1+4\,|x_E|^2)^{-\delta} \sum_{|x-x_E|\leqslant|x_E|} |\varphi_E(x)|^2 \geqslant (1+4\,|x_E|^2)^{-\delta}\,(1-C(\varepsilon,\gamma)\,e^{-2(\gamma-\varepsilon)\,|x_E|}) \tag{A.2}$$

where we used $\|\varphi_E\|_{\ell^2}=1$. It is a result of [7] that there exists only a finite number of E so that $C(\varepsilon,\gamma)\,e^{-2(\gamma-\varepsilon)\,|x_E|}>1/2$. It is then clear that, for some E-independent constant K, the right hand side of inequality (A.2) is greater than $e^{-\varepsilon\,|x_E|}/K$. ∎

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